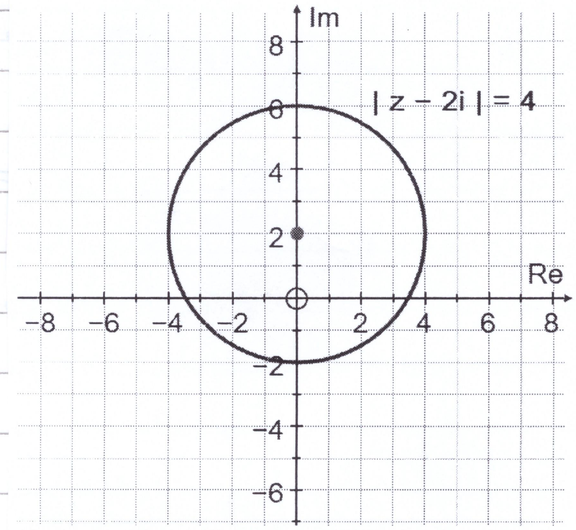
Understanding Pure Mathematics
Sadler and Thorning



Chapter 18 : Algebra II

Exercise 18 F

① $|z - 2i| = 4.$

geometrically : The distance of any complex № z from point 2i is always equal to 4

Algebraically : let $z = x + iy$. Then

$$|z - 2i| = |x + iy - 2i| = 4$$

$$\therefore \quad |x + i(y-2)| = 4 \Rightarrow \sqrt{x^2 + (y-2)^2} = 4$$

$$\therefore \quad x^2 + (y-2)^2 = 16$$

i.e circle centre $(0, 2)$, radius 4.

② $|z - 1 - 3i| = 5$

geometrically : The distance of any complex number z from point $1+3i$ is always equal to 5.

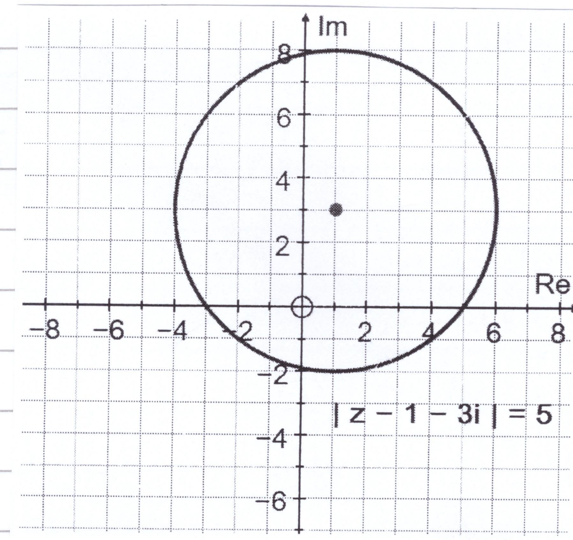
Algebraically : let $z = x + iy$. Then

$|z - 1 - 3i| = |x + iy - 1 - 3i| = 5$, hence $|(x-1) + i(y-3)| = 5$

$$\Rightarrow \sqrt{(x-1)^2+(y-3)^2} = 5$$

$$\therefore \quad (x-1)^2+(y-3)^2 = 25$$

This is a circle centre $(1,3)$
Radius 5



(3) $|z+2-3i| = 4$

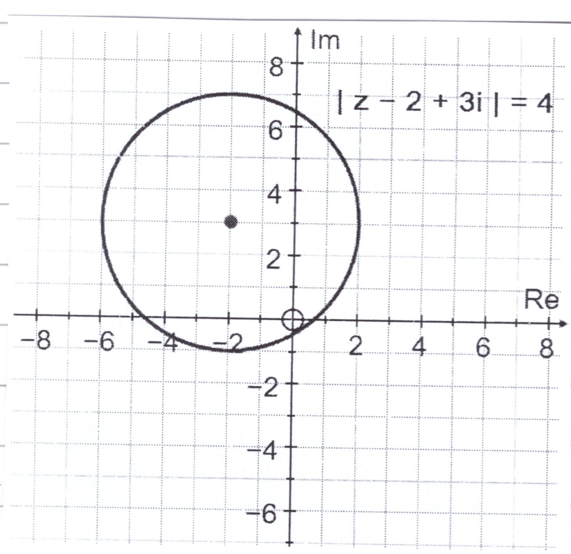
Geometrically: All complex Numbers z, measured from $-2+3i$, have a distance of 4.

Algebraically: Let $z = x+iy$. Then

$$|z+2-3i| = |x+iy+2-3i| = 4$$

$$\therefore \quad |(x+2)+i(y-3)| = 5 \Rightarrow \sqrt{(x+2)^2+(y-3)^2} = 5$$

$$\therefore \quad (x+2)^2+(y-3)^2 = 25$$ is a circle centre $(-2,3)$, Radius 5.



(4) $|z| = |z - 2i|$:

geometrically: The distance from a complex N° z to the origin has to be the same as the distance from the same complex N° to point $2i$.
So the locus $|z| = |z - 2i|$ is the set of points equidistant from $(0,0)$ & $(0,2)$ $\Rightarrow$ The perpendicular bisector of the line joining $(0,0)$ & $(0,2)$

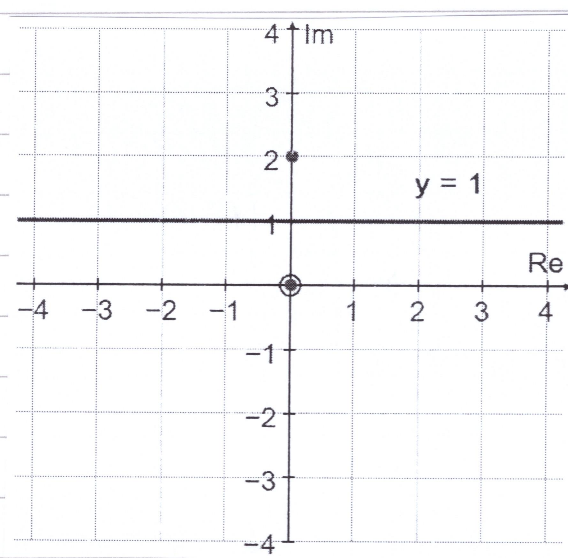
Algebraically: let $z = x + iy$. Then

$|z| = |z - 2i| \Rightarrow |x + iy| = |x + i(y-2)|$

$\therefore \sqrt{x^2 + y^2} = \sqrt{x^2 + (y-2)^2}$

$\therefore x^2 + y^2 = x^2 + (y-2)^2 \Rightarrow y^2 = y^2 - 4y + 4$

$\therefore y = 1$ is a bisector of the line from $(0,0)$ to $(0,2)$



(5) $|z-i| = |z-1|$.

geometrically: The distance of any complex number z from i has to be the same as the distance of that same complex number from 1.
So the locus of $|z-i| = |z-1|$ is the set of points equidistant from $(0,1)$ & $(1,0)$ $\Rightarrow$ a perpendicular bisector to the line joining $(0,1)$ & $(1,0)$.

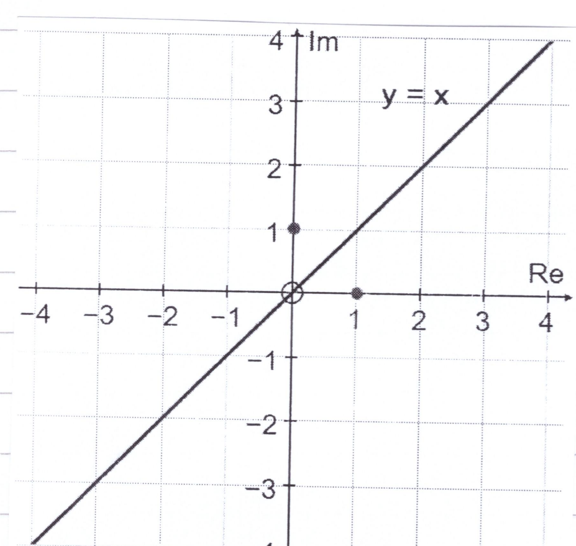
Algebraically: let $z = x+iy$. Then

$$|z-i| = |z-1| \Rightarrow |x+i(y-1)| = |(x-1)+iy|$$

$$\therefore \sqrt{x^2+(y-1)^2} = \sqrt{(x-1)^2+y^2} \Rightarrow x^2+(y-1)^2 = (x-1)^2+y^2$$

$$\therefore x^2+y^2-2y+1 = x^2-2x+1+y^2$$

$$\Rightarrow -2y = -2x \Rightarrow y = x$$



(6) $|z-4+i| = |z-1-2i|$.

geometrically: The distance between any complex N° z and the point $4-i$ is always the same as the distance between that same complex number & the point $1+2i$.
So the locus of $|z-4+i| = |z-1-2i|$ is the set of points equidistant from $(4,1)$ & $(1,2)$ ⇒ the perpendicular bisector of the line joining $(4,1)$ & $(1,2)$.

Algebraically: Let $z = x+iy$. Then

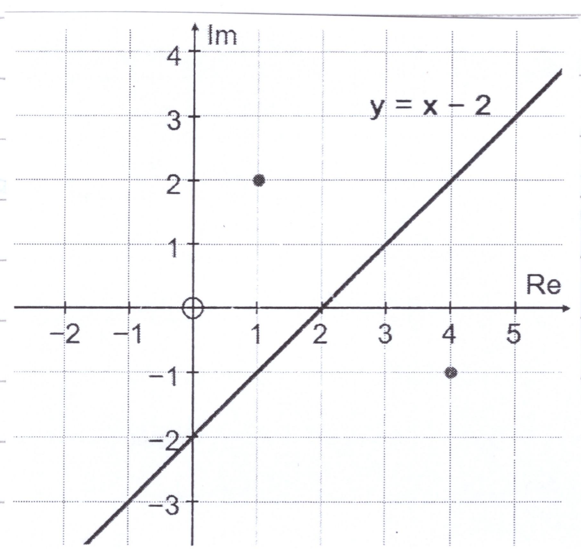
$$|z-4+i| = |z-1-2i| \Rightarrow |(x-4)+i(y+1)| = |(x-1)+i(y-2)|$$

$$\therefore \sqrt{(x-4)^2+(y+1)^2} = \sqrt{(x-1)^2+(y-2)^2}$$

$$\therefore (x-4)^2+(y+1)^2 = (x-1)^2+(y-2)^2$$

$$\Rightarrow x^2-8x+16+y^2+2y+1 = x^2-2x+1+y^2-4y+4$$

hence $-6x+12+6y=0 \Rightarrow y = x-2$



(7) $\underline{z = z^*}$:

Let $z = x + iy$. Then $x + iy = x - iy$

$\Rightarrow 2iy = 0 \Rightarrow y = 0$

i.e The x-axis.

(8) $\underline{z + z^* = 0}$:

Let $z = x + iy$. Then $x + iy + x - iy = 0$

$\Rightarrow 2x = 0 \Rightarrow x = 0$

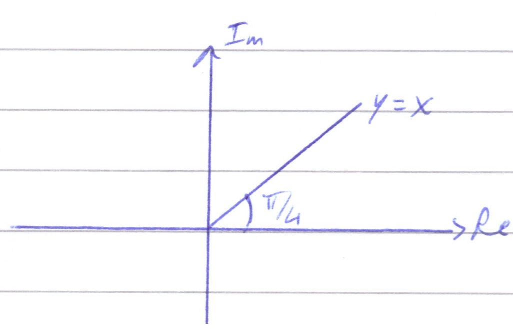
i.e The y-axis.

(9) $\underline{\arg z = \frac{\pi}{4}}$

Geometrically: All complex numbers z are at an angle of $\frac{\pi}{4}$ to the Re-axis. This can only be so if $y = x$.
However, $y = x$ only valid in 1st quadrant, since in 3rd quadrant $\arg z = -3\pi/4 \neq \pi/4$
So the locus of $\arg(z) = \frac{\pi}{4}$ is the half line $y = x$ in quadrant 1 only

Algebraically: $\arg z = \frac{\pi}{4} \Rightarrow \theta = \tan^{-1}\frac{y}{x} = \frac{\pi}{4}$

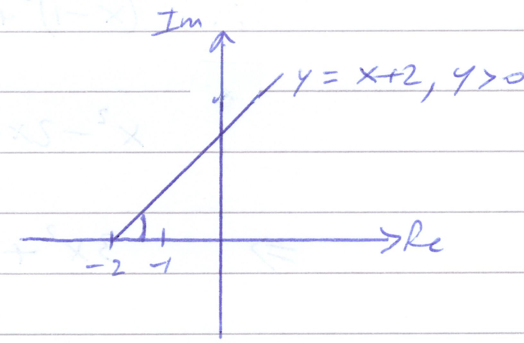
$\therefore \frac{y}{x} = \tan\frac{\pi}{4} = 1 \Rightarrow y = x$ for $y > 0$ only (in order for θ to remain at $\pi/4$ & not $-3\pi/4$)



(10) $\arg(z+2) = \frac{\pi}{4}$

geometric interpretation: The argument / angle of any complex number z, from the pt $(-2, 0)$ is $\pi/4$. The only way this can be so is if all complex numbers lie on a line at $\pi/4$ to the horizontal passing through $(-2, 0)$.

This also implies that the line is a half line i.e a line such that $y > 0$ otherwise any line underneath the x axis will have angle $-3\pi/4$.



Algebraically:

$$\arg(z+2) = \frac{\pi}{4} \Rightarrow \theta = \tan^{-1}\frac{y}{x+2} = \frac{\pi}{4}$$

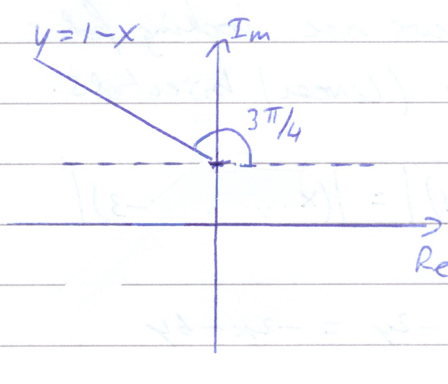
$$\therefore \quad \tan\frac{\pi}{4} = \frac{y}{x+2}$$

$$\Rightarrow y = x+2.$$

(11) $\arg(z-i) = \frac{3\pi}{4}$

geometric meaning: The angle any complex No z makes, from the point $(0, 1)$, is $3\pi/4$. The only way this can be so is if all complex No lie on a straight line from $(0, 1)$, at $\frac{3\pi}{4}$ to the horizontal through $(0, 1)$

Such that $y > 1$



Algebraically: $\arg(z-i) = \frac{3\pi}{4}$

$$\Rightarrow \theta = \tan^{-1}\frac{y-1}{x} = \frac{3\pi}{4}$$

$$\Rightarrow \frac{y-1}{x} = \tan\frac{3\pi}{4} = -1 \ldots$$

(12) Find locus of $|z-1| = 2|z+2-3i|$.

Sol: Let $z = x+iy$. Therefore

$$|x+iy-1| = 2|x+iy+2-3i|$$

$$\Rightarrow |(x-1)+iy| = 2|(x+2)+i(y-3)|$$

$$\therefore \sqrt{(x-1)^2+y^2} = 2\sqrt{(x+2)^2+(y-3)^2}$$

$$\therefore (x-1)^2+y^2 = 4\left((x+2)^2+(y-3)^2\right)$$

$$x^2-2x+1+y^2 = 4(x^2+4x+4+y^2-6y+9)$$

$$\Rightarrow 3x^2+18x+51+3y^2-24y=0$$

$$\therefore x^2+6x+17+y^2-8y=0$$

$$\Rightarrow (x+3)^2-9+(y-4)^2-16+17=0$$

$$\therefore (x+3)^2+(y-4)^2=8$$

Therefore locus is a circle centre $-3+4i$, Radius $\sqrt{8}$.

(13) Each locus is a perpendicular bisector. So we are looking for the point x which is the intersection of these (linear) bisectors.

$$\therefore |z-3-i| = |z-1-3i| \Rightarrow |(x-3)+i(y-1)| = |(x-1)+i(y-3)|$$

$$\therefore (x-3)^2+(y-1)^2 = (x-1)^2+(y-3)^2 \Rightarrow -6x-2y = -2x-6y$$

$\therefore$ $y=x$ is the locus here.

Also $|z+3i| = |z+2-i| \Rightarrow |x+i(y+3)| = |(x+2)+i(y-1)|$

$\therefore \quad x^2 + (y+3)^2 = (x+2)^2 + (y-1)^2$

$\therefore \quad 6y + 9 = 4x - 2y + 5 \Rightarrow 2y = x - 1$ is the other locus

So X (the point of intersection) is

$$2x = x - 1 \Rightarrow x = -1$$
$$\Rightarrow y = -1$$

$\therefore$ point of intersection is $z = -1 - i$.

(14) (a) $|z| = \sqrt{17} \Rightarrow |x + iy| = \sqrt{17}$

$$\therefore \sqrt{x^2 + y^2} = \sqrt{17} \Rightarrow x^2 + y^2 = 17.$$

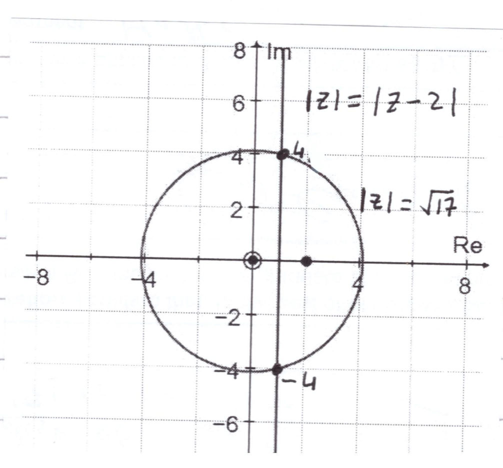
(b) $|z| = |z-2| \Rightarrow |x+iy| = |(x-2)+iy|$

$\therefore \quad x^2 + y^2 = (x-2)^2 + y^2 \Rightarrow -4x + 4 = 0$

$\therefore$ locus is the line $x = 1$

Then for $x = 1$, $1 + y^2 = 17 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$

So points of intersection are $(1, 4)$ and $(1, -4)$



(15) Let $z = x + iy$

Then $|z + 1 - 4i| = |(x+1) + i(y-4)|$

$= \sqrt{(x+1)^2 + (y-4)^2}$

& $|z - 2 - i| = |(x-2) + i(y-1)|$

$= \sqrt{(x-2)^2 + (y-1)^2}$

So we want $(x+1)^2 + (y-4)^2 \geqslant (x-2)^2 + (y-1)^2$

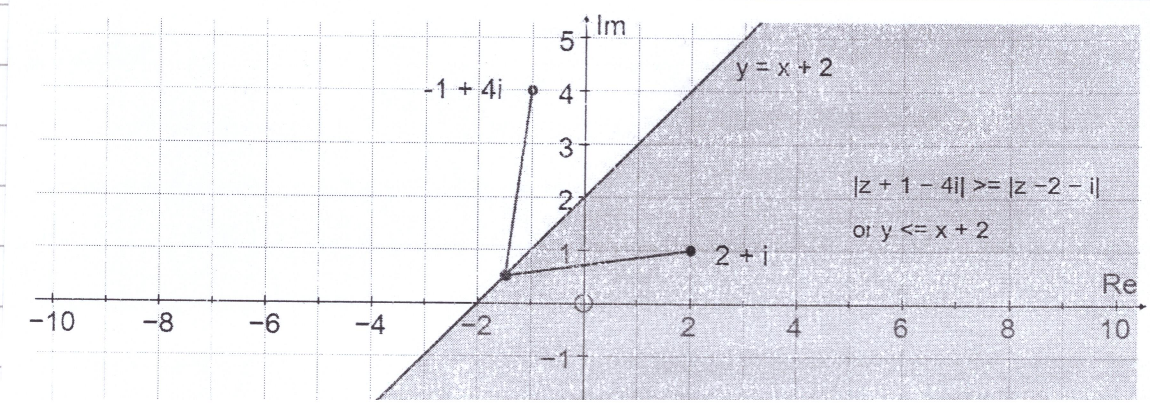
$x^2 + 2x + 1 + y^2 - 8y + 16 \geqslant x^2 - 4x + 4 + y^2 - 2y + 1$

$\Rightarrow \quad 2x - 2y + 4 \geqslant 0 \quad \Rightarrow \quad y \leq x + 2$

OR $|z + 1 - 4i| = |z - 2 - i|$ Represents The perpendicular bisector to the line between $(-1, 4)$ & $(2, 1)$. We want all values / complex N°s s.t.

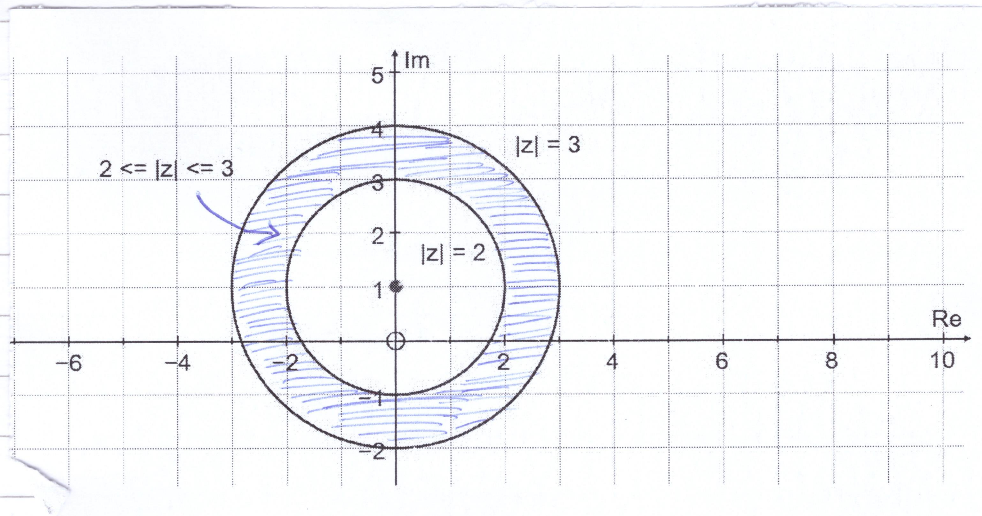
$|z + 1 - 4i| \geqslant |z - 2 - i|$

i.e all complex N°s whose distance from $-1 + 4i$ is greater Than The distance from $2 + i$



(16) For $2 \le |z - i| \le 3$, Note That $|z - i| = k$ is a circle centre $(0, 1)$, or $+i$, with radius k.

In our case we want all complex N°s whose distance From i is $\geqslant 2$ & ≤ 3. The Region is as shaded below:



(17) For $|z| \le 4$ we have a circle centre $(0,0)$ & Radius 4. So we want The Region inside The circle, i.e. all complex N°s whose distance is ≤ 4 From $(0,0)$

For $|z - 3 - i| \geqslant |z - 3 - 5i|$ we have a perpendicular bisector to The line joining $3+i$ & $3+5i$, & we want all complex N°s which lie above The perpendicular bisector.

So The Region where both $|z| \le 4$ and $|z - 3 - i| \geqslant |z - 3 - 5i|$ is The Region Above the Perpendicular bisector & below $|z| \le 4$